# Enhancing Students' Written Mathematical

FUSE/THINKSTOCK

# Arguments

*Writing in mathematics is complex. To swim in the deep end of mathematical understanding, a peer-review activity helps students support a claim and communicate it clearly while keeping the audience in mind.*

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Students who generate arguments in mathematics classrooms take part in an important practice: reasoning and justification. As a mathematical practice, argumentation involves both conceptual and procedural reasoning to justify a claim and can occur by speaking or by writing. Both of these practices have been endorsed by recent policy (CCSSI 2010; NCTM 2000). Reform curricula reflect these endorsements as shown when students are asked to write mathematical statements to support their case, following prompts using such language as "explain why," "convince," and "justify." Moreover, with the widespread adoption of the Common Core State Standards for Mathematics (CCSSM), students will be expected to construct and critique viable arguments by "build[ing] a logical progression of statements to explore the truth of their conjectures . . . justify their conclusions, communicate them to others, and respond to the arguments of others" (CCSSI 2010, pp. 6–7).

To justify a conjecture, students need to understand which mathematical resources are appropriate to draw from, such as symbols, tables, graphs, pictures, and explanations given in words. This task, which has historically been difficult for students in many disciplines (e.g., Deatline-Buchman and Jitendra 2006; Balacheff 1988; McNeill and Krajcik 2009), can be supported with instruction. Given that students will be expected to write convincing arguments, it is important to identify ways that teachers can support their students in doing so.

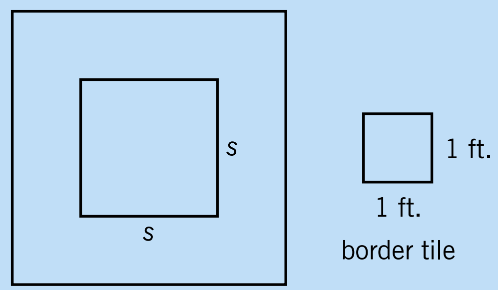
ION POPA

The purpose of this article is to share how one teacher, Ms. Hill, used peer-review activities involving rubrics to explicitly communicate mathematical resources that students could draw from when justifying a claim. She found that helping students understand which type of statements could be used in justification required consistent feedback and ongoing practice. The need for this consistent feedback and practice was especially true for the students described in this article. This case study was conducted in the lowest-tracked eighth-grade algebra class, and most of the students struggled mathematically. However, through peer-review activities, students learned to take an active role in evaluating both their peers' and their own arguments using a rubric, or assessment tool, that listed the criteria for a task (Andrade 2005). The rubric gave students the mathematical structure necessary to assess how thoroughly their peers used various resources to support their conjectures. As a result, students' written arguments became more coherent and convincing because they included acceptable resources in their justifications, made explicit links between their claims and justifications, and realized the importance of communicating clearly.

***Although various mathematical resources can be incorporated into their arguments, students still need to use words to unpack the significance of these resources and clearly link them to their claim.***

**Fig. 1** The Pool problem provided the context that asked for justification and communication.

In-ground pools are often surrounded by borders of tiles. The Custom Pool Company gets orders for square pools of different sizes. For example, the pool illustrated below has a side length of *s* feet and is surrounded by square border tiles. All Custom Pool border tiles measure 1 foot on each side. How many border tiles do you need to surround a square pool?



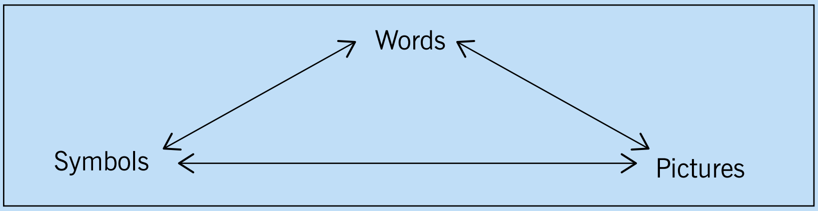

## CREATING OPPORTUNITIES TO ARGUE

One opportunity that Ms. Hill took for providing ongoing practice and feedback for students' arguments came from the Say It with Symbols unit, which is part of the Connected Mathematics Project curriculum (Lappan et al. 2006) in which students explored equivalent expressions. This unit started with the Pool problem, which asked students to generate an expression that would model the number of tiles needed to surround a square pool; they were also asked to justify their expression (see **fig. 1**).

As she introduced this problem, Ms. Hill drew a triangle on the board and labeled the vertices *words, symbols,* and *pictures* (see **fig. 2**). She explained to students that their arguments needed to include each of these three elements, with the bidirectional arrows indicating that these elements were related. With this illustration, Ms. Hill focused students' attention on three mathematical resources that they could use to defend their conjectures.

First, their explanations needed to be in *words*. She explained that the words must consist of more than a set of mathematical steps because "an algorithm alone does not justify," a practice middle school students often use to validate a claim (Bieda and Lepak 2010). In particular, she emphasized the point that students should use words to explain the relationships between their representations and their symbolic expression. By highlighting these relationships, she drew attention to mathematical resources that students were already using, such as pictures, representations, and symbolic expressions, to state a mathematical claim. She also urged them to include these resources

**Fig. 2** This illustration provided a visual of the linking that should occur in arguments.




in their justification. Driscoll (1999) argued that this practice leads to more coherent arguments because it forces students to articulate clear links between their justification and their claim.

Verbal reminders to use words, pictures, and symbols, however, only resulted in modest improvements in students' written justifications. For example, this statement was a typical student response when asked to explain how to calculate the number of tiles for any length pool using words, pictures, and symbols.

> One way you could do it is take how long a side length is, multiply by 4, and then add 4 because you can't forget the corners.

Although this student made a correct claim, there is little by way of the mathematical resources highlighted by Ms. Hill—words, symbols, and pictures—to justify it.

The student's response including "because you can't forget the corners" does not provide a complete, or even mathematical, justification for the claim. Instead, this student simply wrote the expression that he had determined earlier. The explanation does not elaborate on why you "can't forget the four corners," how that fact relates to the expression, or even what this student interpreted as being "corners." Because this example was typical of most students' responses, Ms. Hill decided to magnify the importance of the model in **figure 2** by actively involving her students in peer-review activities. In particular, she wanted to communicate how to use particular mathematical resources—words, pictures, and symbols—to create strong arguments. The peer-review activity also focused students' attention on the importance of clear communication. To assist students in their role as evaluators, she created rubrics that would clearly express her expectations by providing a standard for using words, pictures, and symbols in their explanations.

## PEER-REVIEW ACTIVITIES

After a week of verbally reminding students of the mathematical resources to include in their justifications through constant feedback on their writing and having only modest success, Ms. Hill decided to create task-specific rubrics (Thompson and Senk 1998). Students were to use these rubrics as a guide for giving feedback to their partners (see **fig. 3**). To reinforce the rubrics, she compiled six explanations from the previous day's writing assignment, which asked students to calculate the number of stickers required to cover each face of a string of connecting cubes for any number of cubes. The six explanations included a range of justifications from poorly to moderately supported. With these explanations, she implemented an activity in which students used the rubrics to assess the six arguments' use of mathematical resources to support the claim. **Figure 4** shows an example of how one student used the rubric

**Fig. 3** This rubric assessed students' work calculating the number of stickers required to cover each face of a string of connecting cubes.

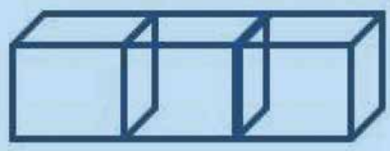
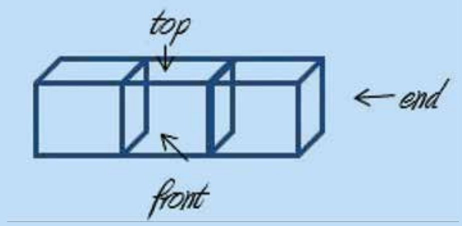
| Rating | 2 | 1 | 0 |
|---|---|---|---|
| Words | An explanation in words about *how* to find the number of stickers *and* "because" statements explaining *why* are provided for each step. | An explanation in words about *how* to find the number of stickers is given, but does not explain *why* it works. | An explanation in words is *not* given. |
| Symbols | An expression showing *how* to find the number of stickers from the cube length is given, *and* each part is labeled with what it represents (*why*). | An expression showing *how* to find the number of stickers, but it is not labeled with what each part represents (*why*). | *No* expression is given. |
| Picture | A labeled picture is used to show how to find the number of stickers. The picture matches the description of words and/or symbols. | A picture is given, but it is not labeled or does not match the description in words or symbols. | *No* picture is given. |

to evaluate three of the six arguments that Ms. Hill assembled and how these examples provided varying levels of explanations.

As arguments were examined by students working in small groups, students expressed difficulty understanding what their peers were trying to communicate, even though they understood the mathematics. Ms. Hill used this opportunity to address the issue of audience with respect to writing convincing explanations. She emphasized that, like the examples used in this exercise, mathematical writers cannot assume that the reader understands what the writer *meant* to say; instead, she instructed her students to consider their audience and to be very clear about how the words, pictures, and symbols are linked to support their claim. In doing so, Ms. Hill provided instruction on how to persuade others of the truth of their claim (Harel and Sowder 1998).

After evaluating the six arguments, students were then asked to create a response that would receive full credit from the rubric, thus incorporating all three mathematical resources that had been the focus in this unit. Before students began working, however, she reminded them to keep their audience in mind. To reinforce attention to their audience, she told students they would share their writing with their partner and that each would use the rubrics to assess each other's writing. In this way, Ms. Hill made paying attention to audience the forefront of students' attention.

**Fig. 4** One student's evaluation of three of the six arguments accompanied the rubrics.

| Explanation | Grade |
|---|---|
| Take the cube length, multiply it by 4 and add 2  | Words: 1<br>Symbols: 0<br>Picture: 1<br>Total: 2 |
| $4x = 2$ | Words: 0<br>Symbols: 1<br>Picture: 0<br>Total: 1 |
| Take the cube length and multiply it by 4 because there are four sides, then add 2 because you need 2 stickers for the ends   | Words: 2<br>Symbols: 0<br>Picture: 2<br>Total: 4 |

## IMMEDIATE RESULTS FROM A PEER REVIEW

Ms. Hill found the peer-review activities (and rubric counterpart) to be an effective way to communicate appropriate mathematical resources for justifying a claim. They were also effective tools that allowed students to identify and connect their thinking about mathematical truths. As a result, students' written arguments became much more coherent and strong. As intended, their arguments began to include mathematical resources with links among words, symbols, and pictures that Ms. Hill had promoted from the beginning of the unit.

I used the rubrics to compare writing assignments the day before and the day after students were given the rubrics. This comparison showed average scores increasing from 1.7 ($n$ = 9) to 5.6 ($n$ = 8) out of 6 possible points. (Only a subset of the larger class was considered because a majority of the students decided not to participate in the study.)

**Figure 5** shows how students' arguments, typical of the rest of the class, became stronger by linking words, symbols, and pictures to represent the thinking that led to their mathematical claim. As was typical of most responses given after viewing the rubric, this explanation included links between the diagram and symbols. The linking statements typically explained the relationship between the mathematical resources that students drew on.

For example, the statement "because there are 2 sides that are width" links the representation to the symbolic expression. Responses of this nature persuade others regarding the validity of the claim because clear, correct statements explicitly connected the mathematical resources to the claim. Moreover, by labeling both the representation and the symbolic expression, the link among words, pictures, and symbols is strong, resulting in a coherent, convincing argument.

## LONG-TERM BENEFITS OF USING RUBRICS

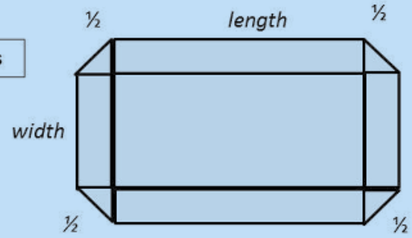
Students' inclusion of words, pictures, and symbols in their arguments

**Fig. 5** This student used words, symbols, and pictures in a successful attempt at following the rubric's categories.

Unfortunately, there are a lot of errors made by new employees. Edit the instructions using the space below so they do a better job explaining ***WHY*** the number sentence works (use a diagram to explain your new explanation).

**Rectangular Pools with Triangle Corners**

*w + w + l + l + 2*
*w = width*
*l = length*
*+2 = four ½ corners added together*




*You would add the width of the pool plus the width, because there are 2 sides that are width and then you add the length of the pool plus the length, because there are 2 sides that are lengths. Then add 2, because on each corner is ½ and if you add ½ plus ½ = 1 and you do that for both top and bottom and that's why you would add 2.*

persisted throughout the unit. A week later, I used the rubrics again to assess whether they had a lasting effect on students' writing. For this writing assignment, students were given the following prompt:

> Crystal says that $2(x - 5)$ is equivalent to $2x - 5$. Explain to Crystal why she is incorrect. Be sure to use words, pictures, and the equation in your explanation.

This task is different from previous writing assignments in two important ways:

1. Students were given abstract expressions for which they needed to determine equivalence where before students were asked to generate expressions and defend them. The fact that students were given expressions rather than being required to generate them made assessing the *symbols* part of the rubric problematic. To receive full credit for this part, students had to use mathematical resources to create a picture and link it to the symbols.
2. The given expressions were not linked to context, yet experience using the rubric encouraged students to tie their explanation to a picture. To provide the link, students used various representations of algebra tiles; a stick represented the variable, dots represented constants, and anything drawn in red represented negative values. The student whose work is shown in **figure 6** used the variable $x$ to represent negative constants to represent $2(x - 5)$ and $2x - 5$.

Despite important differences in the writing prompt, students still performed better than before the peer-review activity, where average scores were 1.7. On average, students scored 4 out of the possible 6 rubric points on this writing assignment, suggesting that the peer-review activity alerted them to the mathematical resources they could use when providing justifications for their argument and the importance of linking the resources.

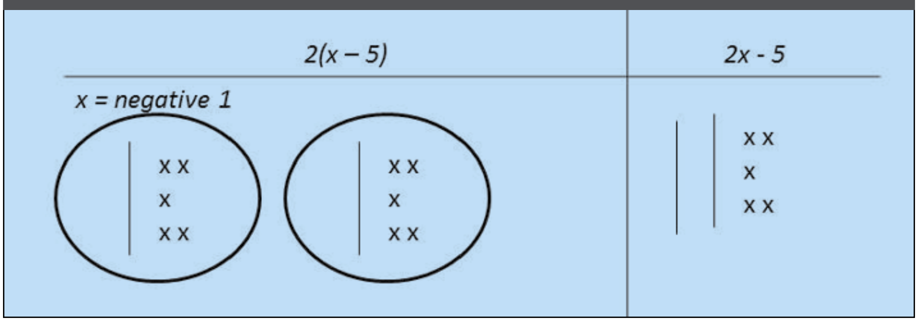
Incidentally, this writing assignment occurred one week after students had engaged in the peer review, and Ms. Hill did not refer to or suggest that students use the rubrics for this writing assignment.

For this class, the peer-review activities helped students develop convincing justifications for their claims by linking words, pictures, and symbols. The inclusion of rubrics needed little explanation and helped make a difficult task clearer, but additional benefits accrued, as well. In particular, students were steered toward using and elaborating on particular mathematical resources and were alerted to the importance of communicating coherently.

## CLARIFYING MATHEMATICAL RESOURCES

Rubrics are used in classrooms for many reasons, including specifying learning objectives and communicating them to students (Andrade 2005). Although Ms. Hill had verbally told students what she expected from their arguments (see **fig. 2**), isolating particular mathematical resources in the rubric seemed to highlight differences among the three tools of words,

**Fig. 6** This example illustrated how a student answered the question, "Is $2(x - 5)$ equivalent to $2x - 5$?"



pictures, and symbols. This allowed students to not only concentrate on one element of reasoning at a time but also link the tools for a stronger argument. For example, to emphasize how the pieces work together, she stated, "Your words, pictures, and symbols need to match. Your words need to describe the expression, and the expression should match the picture" (Ms. Hill, day 5). By reinforcing this instruction with the rubrics, she provided a standard with which students could evaluate their own and each other's written arguments, thus presenting another benefit of rubrics (Carroll 1998).

Although this rubric was task-specific, the overarching principles apply to a wide range of tasks. Using words, in particular, allows students to link mathematical resources that may not be transparent to the reader. Using words to unpack resources may also help students ascertain the truth of their conjecture or find errors in their thinking (Harel and Sowder 1998). Although various mathematical resources can be incorporated into their arguments, students still need to use words to unpack the significance of these resources and clearly link them to their claim.

## COMMUNICATING TO A BROADER AUDIENCE

The exercise that accompanied the rubrics had an unintended, added benefit: making students aware of their audience. Because the instructional activity positioned students to evaluate each other's arguments, the focus was taken from Ms. Hill being the only audience. Attending to audience may be a simple way to focus students' attention on producing convincing arguments that will inform someone other than the teacher (Morgan 1998). Keeping an audience in mind requires students to go into greater depth with their explanation and encourages them to make the links between claim and justification more explicit.

Additionally, as they read and assessed arguments, students became conscious of the difficulty of mathematical communication because this exercise drew their attention to its complexity in written form. Directing the reader's attention to pictures and symbols made the meaning clearer by making specific links among mathematical resources. Although these pictorial representations can help the reader better understand the argument being made, the writer also benefits by being pushed to clearly link the justifications to the claim. Through this activity, Ms. Hill positioned her students as evaluators of mathematical arguments and was able to provide constructive feedback to her students using the rubrics as a guide.

**Keeping an audience in mind required students to go into greater depth with their explanation and encouraged them to make the links between claim and justification more explicit.**

## THE IMPORTANCE OF APPROPRIATE EVIDENCE

Engaging in written arguments is an important mathematical practice that incorporates a thoughtful defense of a claim that draws on appropriate evidence. Drawing students' attention to appropriate evidence is difficult (McNeill and Krajcik 2009; Deatline-Buchman and Jitendra 2006). Accordingly, teachers need to provide students with experiences in argumentative writing and offer tools to help them understand what mathematical resources are appropriate to use when justifying a conjecture. Peer-review activities, like those used in Ms. Hill's class, yielded growth in students' writing scores. Simply telling students what mathematical resources to include in their written arguments did not help them understand how to write a justification. Positioning students as evaluators of mathematical arguments required them to assess the elements of an argument and recognize the importance of coherent communication. This peer-review activity transformed students' arguments as they began to clearly articulate a justification for their claim, drawing on appropriate mathematical resources.

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Any thoughts on this article? Send an e-mail to **mtms@nctm.org**.—*Ed.*

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